2401. The mean is a measure of central tendency, so it is affected by both scaling and translation. The standard deviation and IQR, however, as measures of spread, are affected only by scaling.

(a) True,
(b) False,
(c) True.

2402. Since the intersection we are looking for is near the origin, we know that x is small. Hence, we can use the small-angle approximation $\sin x \approx x$. This gives $5x^2 = 4x + 1$, which is a quadratic. Solving, $x = -0.2, 1$. So, there is an intersection around $x = -0.2$.

To guarantee that $x = -0.2$ to 1dp, we test the values of the function at error bounds:

x	$4x+1$	$5x \sin x$
-0.25	0	0.309
-0.15	0.4	0.112

Since $0 < 4$ and $0.309 > 0.112$, this shows that the curves cross between $x = -0.25$ and $x = -0.15$, so $x = -0.2$ to 1dp.

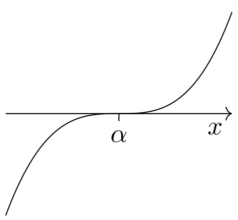
2403. (a) By NIII, the force on each particle has the same magnitude F. So, the acceleration scale factor is the reciprocal of the mass scale factor, i.e. $\frac{1}{2}$. This carries through to the final velocities.
(b) Using $a = \frac{v-u}{t}$, we get $a_1 = \frac{v}{t_0}$ and $a_2 = \frac{v}{2t_0}$.
(c) Relative acceleration is $a_1 + a_2 = \frac{3v}{2t_0}$. Hence, at $t = t_0$, relative displacement is

$$\begin{aligned} s &= \tfrac{1}{2} \cdot \frac{3v}{2t_0} t_0^2 \\ &\equiv \tfrac{3}{4} v t_0. \end{aligned}$$

After this, the relative velocity is $\frac{3}{2}v$. So, in the next $(t - t_0)$ seconds, the particles move a further $\frac{3}{2}v(t - t_0)$ apart. Therefore, the total displacement at time $t \geq t_0$ is

$$\begin{aligned} d &= \tfrac{3}{4} v t_0 + \tfrac{3}{2} v (t - t_0) \\ &\equiv \tfrac{3}{4} v (2t - t_0), \text{ as required.} \end{aligned}$$

2404. The point is a root and a stationary point. The question is whether it is a point of inflection: $g''(\alpha) = 0$ does not guarantee this. However, the second derivative of a cubic is linear, and a (non-zero) linear function must change sign at a root. So, we do have a point of inflection:



2405. Enacting the operator by the quotient rule,

$$\begin{aligned} &\frac{d}{dx}\left(\frac{1}{x+y}\right) = 0 \\ \Longrightarrow\ &\frac{-1 \cdot \left(1 + \frac{dy}{dx}\right)}{(x+y)^2} = 0 \\ \Longrightarrow\ &1 + \frac{dy}{dx} = 0 \\ \Longrightarrow\ &\frac{dy}{dx} = -1, \text{ as required.} \end{aligned}$$

2406. Let $F(x)$ be a function such that $F'(x) = y$. The LHS is now

$$\int_a^b y\,dx + \int_c^d y\,dx = F(b) - F(a) + F(d) - F(c).$$

The RHS is

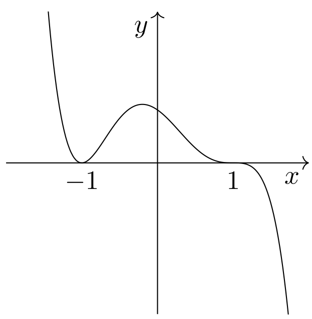
$$\int_a^d y\,dx + \int_b^c y\,dx = F(d) - F(a) + F(b) - F(c).$$

These are the same, so the result is proved.

2407. The original graphs show that:

- the cubic $f(x)$ is positive, with factors $(x+1)$ and $(x-1)^2$,
- the quadratic $g(x)$ is negative, with factors $(x-1)$ and $(x+1)$.

Multiplying these, $f(x)\,g(x)$ is a negative quintic with factors $(x+1)^2$ and $(x-1)^3$. So, it has a double root at $x = -1$ and a triple root at $x = 1$. Hence, the graph $y = f(x)\,g(x)$ is as follows:



2408. Multiplying up, we need $x^2 \equiv A(x+1) + B(x-1)$. But there are no terms in x^2 on the RHS. So, the original identity cannot hold.

2409. (a) Substituting the data points: $240 = A + B$, $180 = A + B \cdot 2^\lambda$, $165 = A + B \cdot 2^{2\lambda}$.

(b) The first equation is $A = 240 - B$. Subbing into the other two, rearranging and taking out a common factor of B,

$$\begin{aligned} -60 &= B(2^\lambda - 1), \\ -75 &= B(2^{2\lambda} - 1). \end{aligned}$$

(c) Dividing gives a quadratic in 2^λ:

$$\begin{aligned}
&\frac{4}{5} = \frac{2^\lambda - 1}{2^{2\lambda} - 1} \\
&\Longrightarrow 4 \cdot 2^{2\lambda} - 4 = 5 \cdot 2^\lambda - 5 \\
&\Longrightarrow 4 \cdot 2^{2\lambda} - 5 \cdot 2^\lambda + 1 = 0 \\
&\Longrightarrow (4 \cdot 2^\lambda - 1)(2^\lambda - 1) = 0 \\
&\Longrightarrow \lambda = 0, -2.
\end{aligned}$$

We reject $\lambda = 0$, which gives $\frac{0}{0}$ in the original equation. So, $\lambda = -2$, $B = 80$, $A = 160$.

(d) The full model is $T = 160 + 80 \times 2^{-2t}$. In the long term, the exponential term decays to zero, and $T \to 160$.

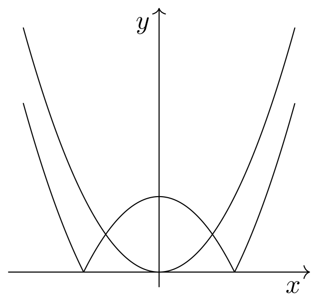
2410. Rearranging to $y \sin u = 0$, we differentiate with respect to y. This gives

$$\sin u + y \cos u \frac{du}{dy} = 0.$$

Dividing both sides by $\cos u$ yields

$$y\frac{du}{dy} + \tan u = 0, \text{ as required.}$$

2411. (a) The graph $y = |x^2 - 1|$ is $y = x^2 - 1$ with its negative portion reflected in $y = 0$:



(b) Both intersections occur on the reflected part of $y = x^2 - 1$, which is $y = 1 - x^2$. So, we solve $1 - x^2 = x^2$, which has roots $x = \pm\sqrt{2}/2$.

2412. We don't need to use the cyclic quadrilateral fact (although we could). We find the midpoints of the diagonals. These are

$$\begin{aligned}
&\left(\tfrac{-5+7}{2}, \tfrac{6-10}{2}\right) = (1, -2) \\
&\left(\tfrac{9-7}{2}, \tfrac{4-8}{2}\right) = (1, -2).
\end{aligned}$$

Since the midpoints of the diagonals coincide, the diagonals bisect each other, as required.

2413. Using the binomial expansion,

$$(2x \pm 1)^3 \equiv 8x^3 \pm 12x^2 + 6x \pm 1.$$

Subtracting the +ve and −ve versions, two terms cancel, leaving $24x^2 + 2 > 98$, which gives $x^2 > 4$. The solution set, therefore, is $(\infty, -2) \cup (2, \infty)$.

2414. (a) This is false. Consider k for which $\text{f}(k) = 0$ but $\text{g}(k) \neq 0$. This value k is neither in P nor in Q, so the set $P \cup Q$ cannot be $\mathbb{R}$.

(b) This is true. There can be no value k for which $\text{f}(k) = 0$ and $\text{f}(k) \neq 0$.

2415. A counterexample is $3, 5, 7$: all three are prime and 5 is the mean of the other two.

2416. Using a compound-angle formula,

$$\begin{aligned}
\sin 15^\circ &= \sin(45^\circ - 30^\circ) \\
&= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\
&= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\
&= \frac{\sqrt{6} - \sqrt{2}}{4}.
\end{aligned}$$

2417. To take (a, b) to $(-a, b)$, we reflect the parabola in the y axis. This means replacing x with $-x$. So, the new parabola is $y = -(-x)^2 + 6(-x) + 4$, which simplifies to $y = -x^2 - 6x + 4$.

— Nota Bene —

If two parabolae are reflection in a line $x = k$, then they may also be thought of as translations by some vector $a\mathbf{i}$. This is only true of a graph which itself has a line of symmetry. If the graphs were cubics, the above technique would not work.

2418. The numerator is $4x \equiv 2(2x - 1) + 2$. Hence,

$$\begin{aligned}
&\int \frac{4x}{2x - 1}\, dx \\
&= \int \frac{2(2x - 1) + 2}{2x - 1}\, dx \\
&= \int 2 + \frac{2}{2x - 1}\, dx \\
&= 2x + \ln|2x - 1| + c, \text{ as required.}
\end{aligned}$$

2419. (a) $[0, 1) \cap (0, 1] = (0, 1)$,

(b) $[0, 2] \setminus [1, 2] = [0, 1)$,

(c) $(0, 1) \cup \{0, 1\} = [0, 1]$.

2420. The third differences are constant. So, the second differences are partial sums of a constant sequence: they are linear. The first differences are partial sums of a linear sequence: they are quadratic. The original terms are then the partial sums of a quadratic sequence: they are cubic.

To summarise: calculating partial sums, which is analogous to integration, raises the power by one each time. □

2421. The area of the rectangle is

$$p\sqrt[n]{p} = p^{1+\frac{1}{n}} = p^{\frac{n+1}{n}}.$$

The shaded area is then given by

$$\begin{aligned}&\int_0^p x^{\frac{1}{n}}\,dx\\ \equiv\ &\left[\tfrac{n}{n+1}x^{\frac{n+1}{n}}\right]_0^p\\ \equiv\ &\tfrac{n}{n+1}p^{\frac{n+1}{n}}.\end{aligned}$$

So, the fraction of the rectangle which is shaded is

$$\frac{\frac{n}{n+1}p^{\frac{n+1}{n}}}{p^{\frac{n+1}{n}}} \equiv \frac{n}{n+1}.$$

Hence, the curve divides the rectangle in the ratio $1:n$, as required.

2422. We can express e as $10^{\log_{10} e}$, which gives

$$\begin{aligned}y &= e^x\\ &\equiv \left(10^{\log_{10} e}\right)^x\\ &\equiv 10^{x\log_{10} e}.\end{aligned}$$

So, consider the transformation

$$y = 10^x \longmapsto y = 10^{x\log_{10} e}.$$

The input x has been replaced by $x\log_{10} e$. Hence, $y = 10^x$ may be transformed to $y = e^x$ by a stretch, scale factor $1/\log_{10} e$, in the x direction. This scale factor may be simplified to $\ln 10$.

— Nota Bene —

This simplification uses the fact that

$$\log_a b \times \log_b a \equiv 1.$$

2423. Solving $\text{f}(x) = -1$,

$$\begin{aligned}&x^4 - 2x^2 + 1 = 0\\ \Longrightarrow\ &(x^2-1)^2 = 0\\ \Longrightarrow\ &x = \pm 1.\end{aligned}$$

Differentiating, $\text{f}'(x) = 4x^3 - 4x$. Evaluating this, $\text{f}'(\pm 1) = \pm 4 \mp 4 = 0$. So, the function f is neither increasing nor decreasing; it is stationary.

2424. The integers differ by 2. So, naming the larger n, $n = m + 2$. This gives

$$m^2 + n^2 = m^2 + (m+2)^2 \equiv 2m^2 + 4m + 4.$$

This has a factor of two. And the remaining factor is $m^2 + 2m + 2 \equiv (m+1)^2 + 1$, which, for $m \in \mathbb{N}$, is greater than 1. So, $m^2 + n^2$ cannot be prime. □

2425. Over a common denominator, the LHS is

$$\frac{1}{(1+\sqrt{x})^4} + \frac{1}{(1-\sqrt{x})^4} \equiv \frac{2x^2 + 12x + 2}{(1-x)^4}.$$

So, the equation is $2x^2 + 12x + 2 = (1-x)^4$, which we can simplify to $x^4 - 4x^3 + 4x^2 - 16x - 1 = 0$. The Newton-Raphson iteration is

$$x_{n+1} = x_n - \frac{x_n^4 - 4x_n^3 + 4x_n^2 - 16x_n - 1}{4x_n^3 - 12x_n^2 + 8x_n - 16}.$$

Running this with $x_0 = 0$ gives $x_1 = -0.0625$, and then $x_n \to -0.0615$ (3sf). Running it with $x_0 = 10$ gives $x_1 = 7.82$ and then $x_n \to 4.01$ (3sf).

2426. The integral formula is

$$\int \text{f}'(x)\big(\text{f}(x)\big)^n\,dx = \frac{1}{n+1}\,\text{f}(x)^{n+1} + c.$$

This is shown by differentiation. By the chain rule,

$$\begin{aligned}&\frac{d}{dx}\left(\tfrac{1}{n+1}\big(\text{f}(x)\big)^{n+1}\right)\\ \equiv\ &\tfrac{1}{n+1}(n+1)\big(\text{f}(x)\big)^n\,\text{f}'(x)\\ \equiv\ &\text{f}'(x)\big(\text{f}(x)\big)^n,\ \text{as required.}\end{aligned}$$

2427. (a) If the string were extensible, the accelerations of the masses could be different. So, we must assume that the string is inextensible.

(b) The equation of motion for the whole system along the string is

$$\begin{aligned}&mg = (M+m)a\\ \Longrightarrow\ &a = \frac{mg}{M+m},\ \text{as required.}\end{aligned}$$

2428. Using a log rule,

$$\begin{aligned}\log_{ab} a^n + \log_{ab} b^n &\equiv \log_{ab} a^n b^n\\ &\equiv \log_{ab}(ab)^n.\end{aligned}$$

By the definition of a logarithm, this is n.

2429. The volume is $V = \frac{1}{3}\pi r^2 h$. We differentiate this implicitly with respect to t, using the product and chain rules:

$$\begin{aligned}&V = \tfrac{1}{3}\pi r^2 h\\ \Longrightarrow\ &\frac{dV}{dt} = \tfrac{1}{3}\pi\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right).\end{aligned}$$

Substituting the various quantities in,

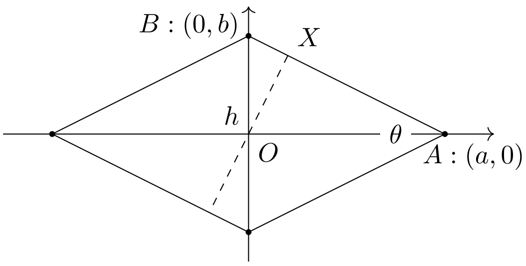
$$\begin{aligned}\frac{dV}{dt} &= \tfrac{1}{3}\pi\left(2x\cdot\tfrac{1}{4}x\cdot x + x^2\cdot\tfrac{1}{3}x\right)\\ &\equiv \tfrac{5\pi}{18}x^3,\ \text{as required.}\end{aligned}$$

2430. The function is well defined where $5x^2 - 3x - 1 > 0$. Solving for the roots of the boundary equation, we get $x = -0.239, 0.839$. Since the domain $[1, \infty)$ lies entirely outside the interval $[-0.239, 0.839]$, the quadratic is positive over this domain, and f is thus well defined.

2431. We break up the fraction by writing the numerator as $3x-1 \equiv (3x-2)+1$, before integrating by the reverse chain rule:

$$\begin{aligned}&\int \frac{3x-1}{3x-2}\,dx \\ &= \int \frac{3x-2+1}{3x-2}\,dx \\ &= \int 1 + \frac{1}{3x-2}\,dx \\ &= x + \tfrac{1}{3}\ln|3x-2| + c.\end{aligned}$$

2432. We note first that, since the interior angles are θ and $(180° - \theta)$, $\sin\theta$ and therefore $\operatorname{cosec}\theta$ has the same value whichever internal angle is chosen. So, let the angle at the right-hand vertex be θ. Adding axes, the situation is:



$\triangle OAX$ gives $\frac{1}{2}h = a\sin\frac{\theta}{2}$. Likewise, $\triangle OBX$ gives

$$\begin{aligned}\tfrac{1}{2}h &= b\sin\left(90° - \tfrac{\theta}{2}\right) \\ &\equiv b\cos\tfrac{\theta}{2}.\end{aligned}$$

Rearranging these,

$$\begin{aligned}a &= \frac{h}{2\sin\frac{\theta}{2}}, \\ b &= \frac{h}{2\cos\frac{\theta}{2}}.\end{aligned}$$

The area is that of four triangles:

$$\begin{aligned}A_{\text{rhom}} &= 4 \times \tfrac{1}{2}ab \\ &\equiv 2ab \\ &= \frac{h^2}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}.\end{aligned}$$

The identity $\sin 2x \equiv 2\sin x\cos x$ gives

$$\begin{aligned}A_{\text{rhom}} &= \frac{h^2}{\sin\theta} \\ &\equiv h^2\operatorname{cosec}\theta, \text{ as required.}\end{aligned}$$

2433. Factorising, we get $x = 0$ or $x^2 + (4 \text{ or } 3)x^2 + 3 = 0$. The discriminants are $\Delta = 4$ or -3.

(a) The quadratic has two roots other than $x = 0$, so the implication doesn't hold.

(b) The quadratic has no real roots, which means the implication holds.

2434. The value of a logarithm is unchanged when one raises both base and input to the same power.

(a) $\log_{27} x^2 \equiv \log_3 x^{\frac{2}{3}} \equiv \frac{2}{3}\log_3 x$.

(b) $\log_{\frac{1}{3}} \dfrac{1}{\sqrt[3]{x}} \equiv \log_3 x^{\frac{1}{3}} \equiv \frac{1}{3}\log_3 x$.

— Nota Bene —

The log rule $\log_a b \equiv \log_{a^n} b^n$ doesn't usually make it onto standard lists of log rules. But it's a useful one. It's the log equivalent of multiplying top and bottom of a fraction by the same thing:

- Multiplying top and bottom of a fraction by the same thing doesn't change its value:

$$\tfrac{6}{2} = \tfrac{60}{20}.$$

"The number you need to multiply 2 by to get 6 is the same as the number you need to multiply 20 by to get 60."

- Raising base and input of a logarithm to the same power doesn't change its value:

$$\log_2 8 = \log_4 64.$$

"The number you need to raise 2 by to get 8 is the same as the number you need to raise 4 by to get 64."

2435. (a) Since the boy exerts 400 N at 60° below the horizontal on the sledge, the sledge must, by NIII, exert 400 N at 60° above the horizontal on the boy. Both of these are in addition to the usual reaction of $R = 25g$. Since the usual reaction exactly balances the boy's weight, the resultant force on the boy is 400 N.
Hence, the acceleration is

$$a = \frac{400}{25} = 16 \text{ ms}^{-2}.$$

The final velocity is

$$\begin{aligned}v &= u + at \\ &= 16 \times 0.25 \\ &= 4 \text{ ms}^{-1}.\end{aligned}$$

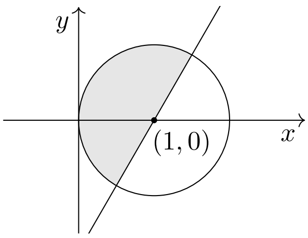
The distance travelled is

$$\begin{aligned}s &= ut + \tfrac{1}{2}at^2 \\ &= \tfrac{1}{2} \times 16 \times 0.25^2 \\ &= \tfrac{1}{2} \text{ m.}\end{aligned}$$

So, the boy travels 50 cm.

(b) The sledge cannot accelerate down through the ground. The only horizontal force on it is $400\cos 60° = 200$ N. Hence, its acceleration is $\frac{200}{10} = 20 \text{ ms}^{-2}$. After take-off, its (horizontal) speed is 5 ms^{-1}.

2436. The scenario is as follows. The centre of the circle is $(1, 0)$, which is also a point on the boundary line.



Since the boundary line goes through the centre of the circle, it is a diameter. Hence, the shaded region is a semicircle, so the probability is $\frac{1}{2}$.

2437. The even and odd cases are different. Knowing first and second term fixes the common ratio r, thus the entire sequence. But knowing the first and third term only fixes r^2, so the hundredth term could be positive or negative.

(a) Yes,
(b) No,
(c) Yes,
(d) No.

2438. The centre of the circle must be equidistant from the three vertices A, B, C. Hence, it must lie on the perpendicular bisector of A and B, and also the perpendicular bisector of B and C. Since the sides of the triangle cannot be parallel, neither can these bisectors be, so there is exactly one point where they cross. □

——— Nota Bene ———

This point is the *circumcentre* of the triangle.

2439. (a) The first ten terms approximate the LHS as

$$\begin{aligned}&1^{-1} + 2^{-2} + 3^{-3} + ... + 10^{-10}\\ &= 1.2913 \text{ (4dp)}.\end{aligned}$$

(b) The trapezium rule approximates the RHS as

$$\begin{aligned}&\tfrac{1}{8}\left(0^{-0} + 2\left(\tfrac{1}{4}^{-\frac{1}{4}} + \tfrac{1}{2}^{-\frac{1}{2}} + \tfrac{3}{4}^{-\frac{3}{4}}\right) + 1^{-1}\right)\\ &= 1.2673 \text{ (4dp)}.\end{aligned}$$

2440. The sum of the first 100 integers is

$$\tfrac{1}{2} \cdot 100 \cdot 101 = 5050.$$

The sum of those that are multiples of 5 is

$$\begin{aligned}&5 + 10 + ... + 100\\ &= 5(1 + 2 + ... + 20)\\ &= \tfrac{5}{2} \cdot 20 \cdot 21\\ &= 1050.\end{aligned}$$

So, the sum of those which are not multiples of 5 is $5050 - 1050 = 4000$.

2441. By the chain rule,

$$\frac{dy}{dx} = -24\cos^2 x \sin x.$$

Evaluating at $x = \frac{\pi}{3}$, we get $m = -3\sqrt{3}$. The y coordinate is 0. So, the equation of the tangent is $y = \sqrt{3}(\pi - 3x)$.

2442. (a) The position vectors are

$$\begin{aligned}\mathbf{p} &= \tfrac{1}{2}(\mathbf{b} + \mathbf{c}),\\ \mathbf{q} &= \tfrac{1}{2}(\mathbf{c} + \mathbf{a}),\\ \mathbf{r} &= \tfrac{1}{2}(\mathbf{a} + \mathbf{b}).\end{aligned}$$

(b) i. The position vector of X is

$$\begin{aligned}\overrightarrow{OX} &\equiv \overrightarrow{OA} + \overrightarrow{AX}\\ &= \overrightarrow{OA} + \lambda\overrightarrow{AP}\\ &= \mathbf{a} + \lambda\big(\tfrac{1}{2}(\mathbf{b} + \mathbf{c}) - \mathbf{a}\big)\\ &\equiv (1 - \lambda)\mathbf{a} + \tfrac{1}{2}\lambda(\mathbf{b} + \mathbf{c}).\end{aligned}$$

ii. By symmetry, the position vector of Y is

$$\overrightarrow{OY} = (1 - \mu)\mathbf{b} + \tfrac{1}{2}\mu(\mathbf{a} + \mathbf{c}).$$

(c) For X and Y to coincide, we need $\overrightarrow{OX} = \overrightarrow{OY}$:

$$(1 - \lambda)\mathbf{a} + \tfrac{1}{2}\lambda(\mathbf{b} + \mathbf{c}) = (1 - \mu)\mathbf{b} + \tfrac{1}{2}\mu(\mathbf{a} + \mathbf{c})$$

Equating coefficients of $\mathbf{a}$ gives $1 - \lambda = \frac{1}{2}\mu$ and of $\mathbf{b}$ gives $1 - \mu = \frac{1}{2}\lambda$. Solving, we get $\lambda = \mu = \frac{2}{3}$.

(d) The argument in (c) also works with Z on CR. Hence, there must be a single point $\frac{2}{3}$ of the way along all three lines AP, BQ, CR. So, the three lines are concurrent. QED.

2443. The basic sinusoids $\sin(x)$ and $\cos(x)$ have period 2π. The transformed periods are as follows:

(a) $2\pi \times \frac{1}{3} = \frac{2}{3}\pi$.
(b) The individual waves have periods π and $2\pi/5$, so the full wave has period $\text{lcm}(\pi, 2\pi/5) = 2\pi$.
(c) The individual waves have periods $\pi/2$ and $\pi/4$, so the full wave has period $\text{lcm}(\pi/2, \pi/2) = \pi/2$.

2444. The quotient rule formula gives

$$\begin{aligned}&\frac{d}{dx}\left(\frac{1 + \sqrt{x}}{1 - \sqrt{x}}\right)\\ &\equiv \frac{\big(1 + \sqrt{x}\big)'\big(1 - \sqrt{x}\big) - \big(1 + \sqrt{x}\big)\big(1 - \sqrt{x}\big)'}{\big(1 - \sqrt{x}\big)^2}.\end{aligned}$$

Finding the individual derivatives, this is

$$\begin{aligned}&\frac{\frac{1}{2}x^{-\frac{1}{2}}\big(1 - \sqrt{x}\big) - \big(1 + \sqrt{x}\big) \cdot -\frac{1}{2}x^{-\frac{1}{2}}}{\big(1 - \sqrt{x}\big)^2}\\ &\equiv \frac{x^{-\frac{1}{2}}}{(1 - \sqrt{x})^2}\\ &\equiv \frac{1}{(1 - \sqrt{x})^2\sqrt{x}}, \text{ as required.}\end{aligned}$$

2445. (a) The RHS is

$$1 - \mathrm{S}(-x) \equiv 1 - \frac{e^{-x}}{e^{-x}+1}.$$

We multiply top and bottom by e^x, then put the terms over a common denominator:

$$\begin{aligned} 1 - \frac{1}{1+e^x} &\equiv \frac{1+e^x-1}{1+e^x} \\ &\equiv \frac{e^x}{e^x+1} \\ &\equiv \mathrm{S}(x). \end{aligned}$$

This is the LHS, proving the identity.

(b) We can integrate by inspection: the integrand is of the form $\mathrm{f}'(x)/\mathrm{f}(x)$. This gives

$$\begin{aligned} \int \mathrm{S}(x)\,dx &= \int \frac{e^x}{e^x+1}\,dx \\ &= \ln|e^x+1| + c. \end{aligned}$$

— Nota Bene —

As ever with integration, if further understanding is needed, differentiate the final result and check that you get the integrand.

2446. We begin with a log law:

$$\begin{aligned} & \ln(\sin x) + \ln(\cos x) + \ln 2 = 0 \\ \Longrightarrow\ & \ln(2\sin x \cos x) = 0 \\ \Longrightarrow\ & 2\sin x\cos x = 1. \end{aligned}$$

Using a double-angle formula, this is

$$\begin{aligned} & \sin 2x = 1 \\ \Longrightarrow\ & 2x = ..., \tfrac{\pi}{2}, ... \\ \therefore\ \ & x = \tfrac{\pi}{4}. \end{aligned}$$

2447. The derivative is $\dfrac{dy}{dx} = -x^{-2}$. So, at $(a, 1/a)$ the tangent is

$$\begin{aligned} & y - \frac{1}{a} = -a^{-2}(x-a) \\ \Longrightarrow\ & y = -\frac{1}{a^2}x + \frac{2}{a}. \end{aligned}$$

The y intercept is $\frac{2}{a}$; the x intercept is at

$$\begin{aligned} & 0 = -\frac{1}{a^2}x + \frac{2}{a} \\ \Longrightarrow\ & x = 2a. \end{aligned}$$

So, the area of $\triangle AOB$ is $1/2 \cdot 2/a \cdot 2a = 2$, which is independent of a, as required.

2448. A special case has been ignored. If $n = 2$, then $(n+1)(n-1) = 3 \times 1$. This is a factorisation, but 3 is nevertheless prime. This is the only exception: for $n > 2$ both factors are greater than 1, and the student's argument holds.

2449. Standard differentiation results give

$$\frac{dy}{dx} = \sec^2 x - \operatorname{cosec}^2 x.$$

Evaluating at $x = \frac{\pi}{4}$, we find that

$$\frac{dy}{dx} = 2 - 2 = 0.$$

This is the gradient of the tangent. An attempt to find the negative reciprocal would yield division by zero. But there is no error. The tangent at $x = \frac{\pi}{4}$ is parallel to the x axis, so the normal is parallel to the y axis. It has equation $x = \frac{\pi}{4}$.

2450. (a) The person's displacement at the top is 1 m, so their initial speed is given by $0 = u^2 - 2g$, which yields $u = \sqrt{2g}$.

(b) Acceleration of person is $\frac{\sqrt{2g}}{0.2} = 11.067$ ms^{-2}. Then NII is $R - 60g = 60 \times 11.076$, which gives the reaction during the jump as 1252 N.

(c) By NIII, the same resultant force must act on the Earth during the jump as on the person. So, the equation of motion for the Earth is

$$\begin{aligned} & 1252 - 60g = 6 \times 10^{24} a_{\mathrm{E}} \\ \Longrightarrow\ & a_{\mathrm{E}} = 1.1068 \times 10^{-22}\ \text{ms}^{-2} \end{aligned}$$

This gives the initial speed of the Earth as $u_{\mathrm{E}} = 4.427 \times 10^{-23}$ ms^{-1}.

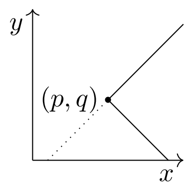
(d) The force on the Earth, while the person is in the air, is $60g$, by NIII. So, the acceleration is $60g/m_{\mathrm{E}} = 9.8 \times 10^{-23}$ ms^{-2}.

(e) The greatest displacement of Earth is given by $0 = u_{\mathrm{E}}^2 - 2a_{\mathrm{E}}s_{\mathrm{E}}$. Rearranging this, we get $s_{\mathrm{E}} = 8.85 \times 10^{-24}$ m (3sf).

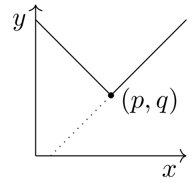
— Nota Bene —

This can be tackled more easily using *momentum*, which is not in the single maths A-level syllabus. This solution is the longer way round via forces, as dictated by the question.

2451. (a) $(y-q) = (x-p)$ (dotted below) is a straight line through (p, q). Applying a mod function to $(y-q)$ requires that $(x-p) \geq 0$, i.e. $x \geq p$. Over this domain, $(y-q)$ can now be positive or negative. So, the graph is



(b) $(y-q)=(x-p)$ is a straight line through (p,q). Applying a modulus function to $(x-p)$ requires $y-q \geq 0$, i.e. $y \geq q$. On this domain, $(x-p)$ can now be positive or negative. So, the graph is



2452. (a) This is true. Since f is increasing everywhere (and has no discontinuities), it can only cross the x axis at most once.

(b) This is false. The function $f(x)=e^{x-10}$ is a counterexample: $e^{x-10}=x$ has two roots.

(c) This is true. Since $y=-x$ has a negative gradient, what applies in (a) applies here. A curve with positive gradient everywhere can only cross $y=-x$ once.

2453. This is binomial, with distribution $B(5, 0.25)$. So,

$$\begin{aligned} &\mathbb{P}(\text{at least 1 above}) \\ &= 1-\mathbb{P}(\text{all below}) \\ &= 1-0.75^5 \\ &= 0.763 \text{ (3sf)}. \end{aligned}$$

2454. The original line has initial position vector $5\mathbf{i}$ and direction vector $\mathbf{i}+3\mathbf{j}$. We transform each to give

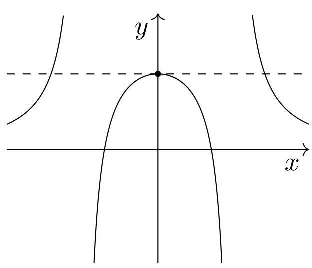
(a) $\mathbf{r}=5\mathbf{j}+t(3\mathbf{i}+\mathbf{j})$,

(b) $\mathbf{r}=-5\mathbf{j}+t(3\mathbf{i}-\mathbf{j})$.

2455. The following function, defined piecewise, is a counterexample:

$$g(x)=\begin{cases} \dfrac{1}{x^2-1}+2, & x \in [-1,1] \\ \dfrac{1}{x^2-1} & x \notin [-1,1]. \end{cases}$$

Its graph is



——— Nota Bene ———

The result is true for polynomials.

2456. Firstly, we solve $1170x^2-389x-165=0$ using the quadratic formula. This gives $x=-11/45, 15/26$. By the factor theorem, $(45x+11)$ and $(26x-15)$ are factors.

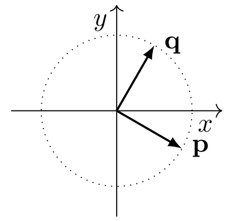
Applying this result to the original quadratic, we know that $(45a+11b)$ and $(26a-15b)$ are factors. Checking that $45 \times 26 = 1170$, no constant factors are needed. The factorisation is

$$\begin{aligned} &1170a^2-389ab-165b^2 \\ &\equiv (45a+11b)(26a-15b). \end{aligned}$$

2457. (a) By the definitions of the sin and cos functions, the magnitudes are 1.

(b) The gradients of $\mathbf{p}$ and $\mathbf{q}$ are $\tan\theta$ and $-\cot\theta$. These are negative reciprocals, so $\mathbf{p}$ and $\mathbf{q}$ are perpendicular.

(c) $\mathbf{p}$ is the position vector of the unit-circle point at $\theta=-\frac{1}{6}\pi$, and $\mathbf{q}$ is perpendicular, at $\theta=\frac{1}{3}\pi$:



(d) We solve simultaneously for the vectors $\mathbf{i}, \mathbf{j}$, by elimination. Multiplying up,

$$\begin{aligned} \cos\theta\mathbf{p} &= \cos^2\theta\mathbf{i}+\sin\theta\cos\theta\mathbf{j}, \\ \sin\theta\mathbf{q} &= -\sin^2\theta\mathbf{i}+\sin\theta\cos\theta\mathbf{j}. \end{aligned}$$

Subtracting, the terms in $\mathbf{j}$ cancel. The first Pythagorean identity gives $\mathbf{i}=\cos\theta\mathbf{p}-\sin\theta\mathbf{q}$. Subbing back in, $\mathbf{j}=-\sin\theta\mathbf{p}+\cos\theta\mathbf{q}$.

2458. Dividing top and bottom by x^2,

$$\begin{aligned} &\lim_{x\to\infty}\frac{1+ax^2}{1+bx^2} \\ \equiv &\lim_{x\to\infty}\frac{\frac{1}{x^2}+a}{\frac{1}{x^2}+b}. \end{aligned}$$

We can now take the limit, with $\frac{1}{x^2}$ tending to zero in both the numerator and the denominator. This gives the value of the limit as a/b.

2459. (a) This is not true. $y=\cos x$ has negative values, which, when put into the modulus function, become positive. In other words, $y=\cos x$ doesn't have the x axis as a line of symmetry.

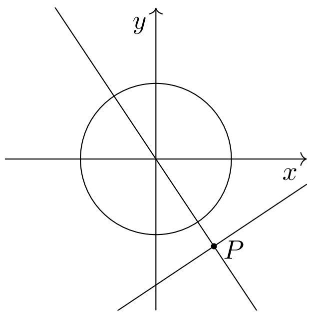
(b) This is true. $y=\cos x$ has the y axis as a line of symmetry (cosine is an even function), so applying the modulus function to its inputs doesn't change its output value.

2460. A prime $p > 3$ cannot leave remainder zero when divided by 3. So, assume, for a contradiction, that p leaves remainder 1 when divided by 3, i.e. that $p = 3k + 1$ for $k \in \mathbb{N}$. Then

$$\begin{aligned} 2p + 1 &= 2(3k + 1) + 1 \\ &\equiv 3(2k + 1). \end{aligned}$$

But this is divisible by 3, which is a contradiction. Hence, a Sophie Germain prime p must leave a remainder of 2 when divided by 3. □

2461. Firstly, we need to show that the line $2x - 3y = 5$ lies outside the unit circle. The closest point on $2x - 3y = 5$ to the origin lies on the normal through O, which is $y = -\frac{3}{2}x$. Solving this with $2x - 3y = 5$, the closest point is $(10/13, -15/13)$, marked P below. This is at a distance $5/\sqrt{13} > 1$ from the origin, so the line lies wholly outside the circle.



Hence, if $2x - 3y > 5$ is true, then (x, y) is below and to the right of the line: outside the circle. So, $2x - 3y > 5 \implies x^2 + y^2 > 1$, as required.

2462. In each answer below, the first set is the range of the denominator. This is then reciprocated to give the range of the function.

(a) $[-1, 3]$ giving $(-\infty, -1] \cup [1/3, \infty)$.
(b) $[0, 4]$ giving $[1/4, \infty)$.
(c) $[1, 5]$ giving $[1/5, 1]$.

2463. This is a quadratic in $x^{\frac{3}{2}}$.

$$\begin{aligned} & x^3 - x^{\frac{3}{2}} - 56 = 0 \\ \implies & (x^{\frac{3}{2}} - 8)(x^{\frac{3}{2}} + 7) = 0 \\ \implies & x^{\frac{3}{2}} = 8, -7. \end{aligned}$$

We reject -7, as $x^{\frac{3}{2}} \geq 0$. So, $x = 8^{\frac{2}{3}} = 4$.

2464. (a) There are 5 ways of filling the first gap. This leaves two gaps, each of which may be filled in 26 ways. This gives $5 \times 26^2 = 3380$ ways.

(b) There remain 5 ways of filling the first gap. The second may be filled in $26 - 4 = 22$ ways, and the third in $26 - 5 = 21$ ways. This gives $5 \times 22 \times 21 = 2310$ ways.

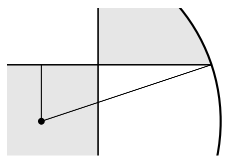
2465. Using the reverse chain rule,

$$\begin{aligned} & \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 - \cos 2x \, dx \\ = & \left[x - \tfrac{1}{2} \sin 2x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ = & \left(\tfrac{\pi}{2} - \tfrac{1}{2} \sin \pi \right) - \left(\tfrac{\pi}{4} - \tfrac{1}{2} \sin \tfrac{\pi}{2} \right) \\ = & \tfrac{\pi}{4} + \tfrac{1}{2}. \end{aligned}$$

2466. This is true.

If the linear equations are scalar multiples of each each, such as $x + y = 3$ and $2x + 2y = 6$, then any (x, y) pair satisfying one automatically satisfies the other. Graphically, both equations are the same line, which contains infinitely many (x, y) points.

2467. Drawing in a radius gives a right-angled triangle, lengths $\frac{1}{2}l$ and $\frac{3}{2}l$.



By Pythagoras,

$$\begin{aligned} r^2 &= \tfrac{1}{4}l^2 + \tfrac{9}{4}l^2 \\ &\equiv \tfrac{5}{2}l^2. \end{aligned}$$

This gives $A = \pi r^2 = \frac{5}{2}\pi l^2$, as required.

2468. By the quotient rule,

$$\frac{dy}{dx} = \frac{100(100 + x^2) - 100x \cdot 2x}{(100 + x^2)^2}.$$

The derivative is zero iff its numerator is zero. So, $100^2 - 100x^2 = 0 \implies x = \pm 10$. Substituting back in, we get two stationary points, at $(\pm 10, \pm 5)$.

2469. The binomial expansion gives

$$\begin{aligned} & \left(1 \pm \sqrt[3]{2}\right)^3 \\ = & 1 \pm 3 \cdot 2^{\frac{1}{3}} + 3 \cdot 2^{\frac{2}{3}} + 2. \end{aligned}$$

When subtracting, the first and third terms cancel, leaving $6\sqrt[3]{2} + 4$.

2470. This is correct. The frictional force on the truck acts inwards, towards the centre of the circle. This is the force that keeps the truck from leaving the roundabout. By NIII, therefore, the frictional force on the road must act outwards.

2471. Differentiating, $g'(x) = 3x^2 + 2kx + k$. For g to be increasing everywhere, we require that, fro all x,

$$3x^2 + 2kx + k > 0.$$

The LHS is a positive quadratic, so it is positive everywhere iff $\Delta < 0$. This is $4k^2 - 12k < 0$. The boundary equation is $4k^2 - 12k = 0$, giving $k = 0, 3$. So, since $4k^2 - 12k$ is a positive quadratic, its value is negative for $k \in (0, 3)$.

2472. The translations associated with $+c$ and $+d$ do not affect areas, so we can consider $y = ae^{px}$ being transformed to $y = be^{qx}$.

The outputs are scaled by b/a, which is a stretch by scale factor b/a in the y direction.

In the inputs, px has been replaced by qx, which is the same as replacing x by $(q/p)x$. This is a stretch by scale factor p/q in the x direction.

Overall, the area scale factor is

$$\text{ASF} = \frac{b}{a} \times \frac{p}{q} \equiv \frac{bp}{aq}.$$

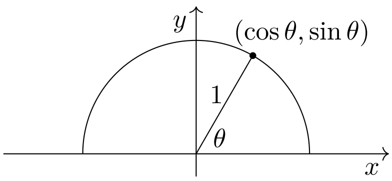
2473. The magnitudes of the accelerations are the same. So, we can form an equation of motion for the whole system. The tensions are an internal force pair and cancel, leaving

$$\begin{aligned} & m_2 g - m_1 g = (m_1 + m_2)a \\ \Longrightarrow\ & (m_2 - m_1)g = (m_1 + m_2)a \\ \Longrightarrow\ & a = \frac{m_2 - m_1}{m_1 + m_2} g, \text{ as required.} \end{aligned}$$

2474. A point on the unit circle is $(\cos\theta, \sin\theta)$. So, for $x \in [-1, 1]$, we have $\arccos x = \theta$, where $\theta \in [0, \pi]$. This gives

$$\sin(\arccos x) = y.$$

Hence, the graph is made up of all (x, y) points of the form $(\cos\theta, \sin\theta)$ for $\theta \in [0, \pi]$.



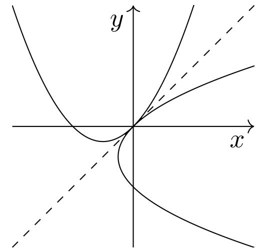
This is the upper unit semicircle.

2475. (a) If $f(x)\,g(x) = 0$, then $f(x) = 0$ or $g(x) = 0$. So, $x \in A \cup B$.

(b) If $fg(x) = 0$, then $g(x) \in A$. We do not have enough information to solve this.

(c) If $f(x)/g(x) = 0$, then $f(x) = 0$. And, since there is nothing in the solution set of both A and B, the fraction is well defined for every element of A. So, the solution set is A.

2476. The curves are reflections in $y = x$. So, we look for points of tangency on $y = x$. Solving $x^2 + x = x$ gives $x^2 = 0$. Since $x = 0$ is a double root, we know that this is a point of tangency between $y = x^2 + x$ and $y = x$. Hence, it is also a point of tangency between $y = x^2 + x$ and $x = y^2 + y$:

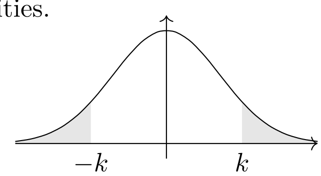


2477. (a) The derivative is $\frac{dy}{dx} = 3x^2 + x$. So, $m = 14$ at $(2, 13)$. The equation of the tangent line is $y - 13 = 14(x - 2)$, i.e. $y = 14x - 15$.

(b) Substituting for y, $x^3 + 2x + 1 = 14x - 15$, which simplifies to $x^3 - 12x + 16 = 0$.

(c) We can use the factor theorem. Since $x = 2$ must be a (double) root of the above equation, $(x - 2)$ must be a (double) factor. This gives $(x - 2)^2(x + 4) = 0$. So, the coordinates of P are $(-4, -71)$.

2478. (a) This is due to the symmetry of the normal distribution. The shaded areas represent the probabilities.



(b) Using the sketch above, the condition $X^2 > k^2$ restricts the possibility space to the shaded area. This gives

i. $\mathbb{P}\big(0 < X < k \mid X^2 > k^2\big) = 0$,
ii. $\mathbb{P}\big(X < 0 \mid X^2 > k^2\big) = \frac{1}{2}$.

2479. The functions are quadratic, so the first derivatives are linear. Considering $y = f'(x)$ and $y = g'(x)$ as straight lines, we know that both pass through $\big(p, f'(p)\big)$ and the distinct point $\big(q, f'(q)\big)$. Since there is only one line through any two points, the derivatives must be identical: $f'(x) \equiv g'(x)$.

Integrating this, $f(x) + c_1 = g(x) + c_2$. We combine the constants to get $f(x) - g(x) = c$, as required.

2480. (a) The mod function doesn't affect the outputs, and all positive inputs are still available, so the range is the same as that of the normal sine function: $[-1, 1]$.

(b) The mod function renders the outputs of the sine function positive, converting the range of the sine function, which is $[-1, 1]$, into $[0, 1]$.

2481. Using the quotient rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2-p)'(x^2-q)-(x^2-p)(x^2-q)'}{(x^2-q)^2}\\ &\equiv \frac{2x(x^2-q)-(x^2-p)2x}{(x^2-q)^2}\\ &\equiv \frac{2x(p-q)}{(x^2-q)^2}.\end{aligned}$$

Since the numerator has a factor of x, it is zero as long as the fraction is well defined. Hence, the only restriction is $q \neq 0$.

2482. $\cos^2 x = 1 - \sin^2 x$ gives a quadratic in $\sin x$:

$$\begin{aligned}&\sin x - 1 = 4\cos^2 x\\ \Longrightarrow\ &\sin x - 1 = 4 - 4\sin^2 x\\ \Longrightarrow\ &4\sin^2 x + \sin x - 5 = 0\\ \Longrightarrow\ &(4\sin x + 5)(\sin x - 1) = 0.\end{aligned}$$

The first factor has no roots, as $-5/4$ is outside the range of the sine function. So, $\sin x = 1$, which holds at $x = 90°$, plus any multiple of 360°. In set notation, this is $\{x : x = 90° + 360°n, n \in \mathbb{Z}\}$.

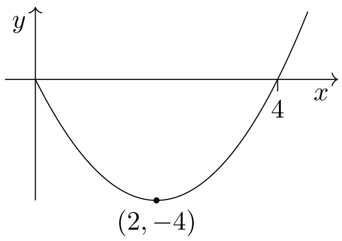
2483. The minimum and maximum horizontal speeds are 19.6 and 20.4 ms^{-1}. For each, the vertical data are $s = -2.5$, $a = -g$, $u = 0$, giving $-2.5 = -4.9t^2$, so time of flight is $t = 5/7$. Multiplying by horizontal speed, $d_{\text{min}} = 14$ m and $d_{\text{max}} = 14.57$ m (2dp).

2484. All $4! = 24$ orders of x_1, x_2, x_3, x_4 are equally likely. So, the probability is $\frac{1}{24}$.

— Nota Bene —

The probability that e.g. $x_1 = x_2$ is zero, as x_1 and x_2 are values chosen from infinitely many.

2485. The vertex of $y = x^2 - 4x$ is at $(2, -4)$. Also we know that $\text{g}(0) = 0$ and $\text{g}(4) = 0$. The sketch is



(a) For $k \in (0, 2)$, the domain does not include the vertex and 0 is the maximum, so the range is $\{y \in \mathbb{R} : \text{g}(k) < y < 0\}$, which is $(k^2 - 4k, 0)$.
(b) For $k \in [2, 4)$, the domain does include the vertex, and 0 is the maximum, so the range is $\{y \in \mathbb{R} : -4 \leq y < 0\}$ or $[-4, 0)$.
(c) For $k \in [4, \infty)$, the domain does include the vertex, and $\text{g}(k)$ is the maximum, so the range is $\{y \in \mathbb{R} : -4 \leq y < \text{g}(k)\}$ or $[-4, k^4 - 4k)$.

2486. (a) Quoting the standard result, $u_n = 5 \times \frac{3}{2}^{n-1}$.
(b) We require $u_{n+1} - u_n > 1000$. The boundary equation is

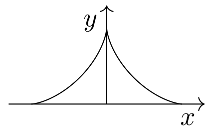
$$\begin{aligned}&5 \times \tfrac{3}{2}^n - 5 \times \tfrac{3}{2}^{n-1} = 1000\\ \Longrightarrow\ &\tfrac{3}{2}^{n-1}(5 \times \tfrac{3}{2} - 5) = 1000\\ \Longrightarrow\ &n - 1 = \log_{\frac{3}{2}} 400\\ \Longrightarrow\ &n = 15.776...\end{aligned}$$

So, 16 is the smallest such value of n.

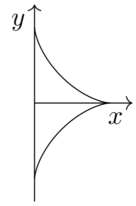
2487. (a) We write $\text{f}(x) = x^x \equiv (e^{\ln x})^x \equiv e^{x \ln x}$.
(b) Differentiating by the chain and product rules,

$$\begin{aligned}\frac{dy}{dx} &= e^{x\ln x}(x \ln x)'\\ &\equiv e^{x\ln x}\big(\ln x + x \cdot \tfrac{1}{x}\big)\\ &\equiv x^x(\ln x + 1).\end{aligned}$$

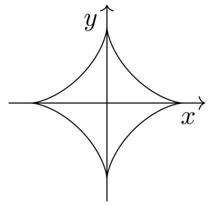
2488. (a) For $y = \text{g}(|x|)$, x can also be negative:



(b) For $|y| = \text{g}(x)$, y can also be negative:



(c) For $|y| = \text{g}(|x|)$, x and y can also be negative:



2489. Using the first Pythagorean trig identity, we have a quadratic in $\sin x$:

$$\begin{aligned}&6\cos^2 x + \sin x - 5 = 0\\ \Longrightarrow\ &6(1 - \sin^2 x) + \sin x - 5 = 0\\ \Longrightarrow\ &6\sin^2 x - \sin x - 1 = 0\\ \Longrightarrow\ &(2\sin x - 1)(3\sin x + 1) = 0\\ \Longrightarrow\ &\sin x = \tfrac{1}{2}, -\tfrac{1}{3}.\end{aligned}$$

Each gives two roots in the given domain. The solution is $x = 30°, 150°, 199.5°, 340.5°$.

2490. Writing longhand and using the chain rule,

$$\begin{aligned}\frac{dx_1}{dx_5} &\equiv \frac{dx_1}{dx_2}\cdot\frac{dx_2}{dx_3}\cdot\frac{dx_3}{dx_4}\cdot\frac{dx_4}{dx_5}\\ &= 1\cdot 2\cdot 3\cdot 4\\ &= 24.\end{aligned}$$

2491. The LHS is

$$\begin{aligned}&\int_0^k (2x-1)^2\,dx\\ \equiv\ &\left[\tfrac{1}{6}(2x-1)^3\right]_0^k\\ \equiv\ &\tfrac{1}{6}(2k-1)^3+\tfrac{1}{6}\\ \equiv\ &\tfrac{4}{3}k^3-2k^2+k.\end{aligned}$$

The RHS is

$$\begin{aligned}&\int_k^4 4(x-2)^3+6\,dx\\ \equiv\ &\Big[(x-2)^4+6x\Big]_k^4\\ \equiv\ &(16+24)-\big((k-2)^4+6k\big)\\ \equiv\ &24+26k-24k^2+8k^3-k^4.\end{aligned}$$

So, the full equation is

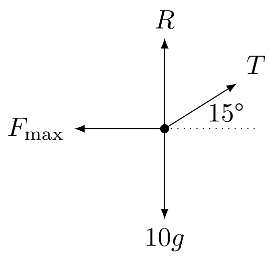
$$\begin{aligned}&\tfrac{4}{3}k^3-2k^2+k=24+26k-24k^2+8k^3-k^4,\\ \Longrightarrow\ &k^4-\tfrac{20}{3}k^3+22k^2-25k-24=0.\end{aligned}$$

Using a calculator's polynomial solver, we get a negative root, which we reject as k is positive, and $k=3$.

2492. The area is $A=\sin t\cos t$, which a double-angle formula converts to $A=\frac{1}{2}\sin 2t$. Differentiating, $\frac{dA}{dt}=\cos 2t$. Setting this to zero so that the area is stationary, $2t=\frac{\pi}{2}$. This gives $t=\frac{\pi}{4}$.

2493. To transform the inputs, we replace x by $x+a$; this is a translation by $-a\mathbf{i}$. To transform the outputs, we add b; this is a translation by $b\mathbf{j}$. So, the overall transformation is translation by vector $-a\mathbf{i}+b\mathbf{j}$.

2494. The force diagram is



Vertical equilibrium gives

$$\begin{aligned}&R+T\sin 15°=10g\\ \Longrightarrow\ &R=10g-T\sin 15°.\end{aligned}$$

Since the sledge is moving, friction is at

$$F_{\text{max}}=\mu R=g-\frac{\sin 15°}{10}T.$$

If the child is pulling with minimal force, then the horizontal forces are also balanced, so

$$\begin{aligned}&T\cos 15°=g-\frac{\sin 15°}{10}T\\ \Longrightarrow\ &T=\frac{g}{\cos 15°+\frac{\sin 15°}{10}}\\ &\quad=9.88\text{ N (3sf)}.\end{aligned}$$

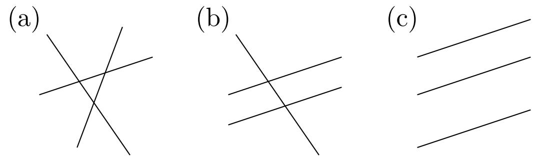
2495. The lines have no simultaneous solution points, so they are not concurrent. This leaves three options:

(a) none are parallel, giving $n=3$,

(b) two are parallel, giving $n=2$,

(c) three are parallel, giving $n=0$.

The sketches are as follows:



2496. The identity is automatically true when $x\geq 0$. When $x<0$, we are told that $\mathrm{g}(|x|)$, which is $\mathrm{g}(-x)$, is equal to $\mathrm{g}(x)$. This means the graph has the y axis as a line of symmetry.

2497. Multiplying out and differentiating,

$$\begin{aligned}&y=\tfrac{1}{5}e^{2x}+\tfrac{4}{5}e^{-3x}\\ \Longrightarrow\ &\tfrac{dy}{dx}=\tfrac{2}{5}e^{2x}-\tfrac{12}{5}e^{-3x}.\end{aligned}$$

Substituting for y and $\frac{dy}{dx}$, the LHS is

$$\begin{aligned}&\frac{dy}{dx}+3y-e^{2x}\\ \equiv\ &\tfrac{2}{5}e^{2x}-\tfrac{12}{5}e^{-3x}+\tfrac{3}{5}e^{2x}+\tfrac{12}{5}e^{-3x}-e^{2x}\\ \equiv\ &0.\end{aligned}$$

Therefore, the given curve satisfies the DE.

2498. A quartic can have precisely three roots when when one is a double root. A counterexample to the claim is $(x-1)^2(x-2)(x-3)=0$.

2499. Let $P=x^3+y^3$; we are trying to maximise P. For $y\geq 0$, we have $y=\sqrt{1-x^2}$. So, we can express P in terms of x as $P=x^3+(1-x^2)^{\frac{3}{2}}$. Looking for a maximum, we set the derivative to zero:

$$\begin{aligned}&\frac{dP}{dx}=3x^2-3x\big(1-x^2\big)^{\frac{1}{2}}=0\\ \Longrightarrow\ &x^2=x\big(1-x^2\big)^{\frac{1}{2}}\\ \Longrightarrow\ &x^4=x^2(1-x^2)\\ \Longrightarrow\ &x^2(2x^2-1)=0.\end{aligned}$$

This yields $x = 0, \pm 1/\sqrt{2}$, giving $P = 1, 1/\sqrt{2}$. The latter is less than 1. So, the maximum value of $x^3 + y^3$ is 1, as required.

——— Alternative Method ———

Since both terms in the LHS of $x^2 + y^2 = 1$ are non-negative, we know that $x^2, y^2 \in [0, 1]$. Raising a number in $[0, 1]$ to the power $3/2$ cannot increase it. Hence, $x^3 \leq x^2$ and $y^3 \leq y^2$. Combining these,

$$x^3 + y^3 \leq x^2 + y^2.$$

The value of $x^2 + y^2$ is 1, so the value of $x^3 + y^3$ is bounded above by 1. This bound is attainable, with $x = 1$, $y = 0$. Hence, the maximum value of $x^3 + y^3$ is 1, as required.

2500. Each is an instance of the chain rule. The latter is generally known as implicit differentiation, but it is exactly the same process as the former.

(a) True,

(b) True.

——— End of 25th Hundred ———